

# RATIONALITY OF ALGEBRAIC CYCLES OVER FUNCTION FIELD OF $\mathbf{SL}_1(A)$ -TORSORS

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**ABSTRACT.** In this note we prove a result comparing rationality of algebraic cycles over the function field of a  $\mathbf{SL}_1(A)$ -torsor for a central simple algebra  $A$  and over the base field.

**Keywords:** Chow groups, central simple algebras, principal homogeneous spaces.

## CONTENTS

1. Introduction	1
2. Preliminaries	2
3. Proof of the result	4
4. Exceptional projective homogeneous varieties	5
References	6

## 1. INTRODUCTION

Let  $A$  be a central simple algebra over a field  $F$  and let  $\mathrm{Nrd} : A^\times \rightarrow F^\times$  be the reduced norm homomorphism. We recall that the homomorphism  $F^\times \rightarrow H^1(F, \mathbf{SL}_1(A))$ , associating to  $c \in F^\times$  the  $\mathbf{SL}_1(A)$ -torsor  $X_c$  given by the equation  $\mathrm{Nrd} = c$ , is surjective (with kernel  $\mathrm{Nrd}(A^\times)$ ) – see [7, Proposition 2.7.3] for instance.

The main purpose of this note is to prove the following theorem dealing with rationality of algebraic cycles over function field of  $\mathbf{SL}_1(A)$ -torsors.

**Theorem 1.1.** *Let  $A$  be a central simple algebra of prime degree  $p$  over a field  $F$  and let  $X$  be a  $\mathbf{SL}_1(A)$ -torsor. Then*

(i) *for any equidimensional  $F$ -variety  $Y$ , the change of field homomorphism*

$$\mathrm{CH}(Y) \rightarrow \mathrm{CH}(Y_{F(X)}),$$

*where  $\mathrm{CH}$  is the integral Chow group, is surjective in codimension  $< p + 1$ .*

(ii) *it is also surjective in codimension  $p + 1$  for a given  $Y$  provided that the variety  $X_{F(\zeta)}$  does not have any closed point of prime to  $p$  degree for each generic point  $\zeta \in Y$ .*

The method of proof mainly relies on the following statement. This proposition is a version of the result [3, Lemma 88.5] slightly altered to fit our situation (see also the proof of [8, Proposition 2.8]).

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**Proposition 1.2** (Karpenko, Merkurjev). *Let  $X$  be a smooth variety, and  $Y$  an equidimensional variety. Given an integer  $m$  such that for any nonnegative integer  $i$  and any point  $y \in Y$  of codimension  $i$  the change of field homomorphism*

$$\mathrm{CH}^{m-i}(X) \longrightarrow \mathrm{CH}^{m-i}(X_{F(y)})$$

*is surjective, the change of field homomorphism*

$$\mathrm{CH}^m(Y) \longrightarrow \mathrm{CH}^m(Y_{F(X)})$$

*is also surjective.*

The proof of Theorem 1.1 is given in Section 3. In Section 4, we describe how this theorem can be related to a similar result dealing with rationality of algebraic cycles over function field of projective homogeneous varieties under some groups of exceptional type.

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## 2. PRELIMINARIES

**2.1. Topological filtration and Chow groups.** For any smooth variety  $X$  over a field  $F$  (in this paper, an  $F$ -variety is a separated scheme of finite type over  $F$ ), one can consider the topological filtration on the Grothendieck ring  $K_0(X)$ , whose term of codimension  $i$  is given by

$$\tau^i(X) = \langle [\mathcal{O}_Z] \mid Z \hookrightarrow X \text{ and } \mathrm{codim}(Z) \geq i \rangle,$$

where  $[\mathcal{O}_Z]$  is the class in  $K_0(X)$  of the structure sheaf of a closed subvariety  $Z$ . We write  $\tau^{i/i+1}(X)$  for the successive quotients. We denote by  $pr^i$  the canonical surjection

$$\begin{array}{ccc} \mathrm{CH}^i(X) & \twoheadrightarrow & \tau^{i/i+1}(X) \\ [Z] & \mapsto & [\mathcal{O}_Z] \end{array},$$

where  $\mathrm{CH}$  is the integral Chow group. By the Riemann-Roch Theorem without denominators the  $i$ -th Chern class induces an homomorphism in the opposite way  $c_i : \tau^{i/i+1}(X) \rightarrow \mathrm{CH}^i(X)$  such that the composition  $c_i \circ pr$  is the multiplication by  $(-1)^{i-1}(i-1)!$ .

Note that for any prime  $p$ , one can also consider the topological filtration  $\tau_p$  on the ring  $K_0(X)/pK_0(X)$  by replacing  $K_0(X)$  by  $K_0(X)/pK_0(X)$  in the previous definition. In particular, we get that for any  $0 \leq i \leq p$ , the map  $pr_p^i : \mathrm{Ch}^i(X) \rightarrow \tau_p^{i/i+1}(X)$ , where  $\mathrm{Ch}$  is the Chow group modulo  $p$ , is an isomorphism.

**Remark 2.1.** Assume that  $X$  is a  $\mathbf{SL}_1(A)$ -torsor and let  $p$  be a prime. One has  $K_0(X) = \mathbb{Z}$  by the result [14, Theorem A] of I. Panin and consequently, for  $i \geq 1$ , the term  $\tau^i(X)$  is equal to zero. Therefore, for any  $1 \leq i \leq p$ , one has  $\mathrm{Ch}^i(X) = 0$ . Moreover, by the result [17, Theorem 2.7] of A. Suslin, one has  $\mathrm{CH}^i(\mathbf{SL}_p) = 0$  for any  $i \geq 1$ . Hence, for  $A$  of degree  $p$  (then there exists a splitting field of  $A$  of degree  $p$ ), it follows by transfert argument that  $p \cdot \mathrm{CH}^i(X) = 0$  for any  $i \geq 1$ . Therefore, for  $X$  a  $\mathbf{SL}_1(A)$ -torsor, with  $A$  of prime degree  $p$ , one has  $\mathrm{CH}^i(X) = 0$  for any  $1 \leq i \leq p$ . Note that, by Proposition 1.2, this gives Theorem 1.1(i) already.

**2.2. Brown-Gersten-Quillen spectral sequence.** For any smooth variety  $X$  and any  $i \geq 1$ , the epimorphism  $pr^i$  coincides with the edge homomorphism of the spectral Brown-Gersten-Quillen structure  $E_2^{i,-i}(X) \Rightarrow K_0(X)$  (see [16, §7]), that is to say

$$pr^i : \mathrm{CH}^i(X) \simeq E_2^{i,-i}(X) \twoheadrightarrow \cdots \twoheadrightarrow E_{i+1}^{i,-i}(X) = \tau^{i/i+1}(X).$$

Assume that  $X$  is a  $\mathbf{SL}_1(A)$ -torsor, with  $A$  of prime degree  $p$ . Then it follows from Remark 2.1 that  $E_{i-1}^{i,-i}(X) = 0$  for  $3 \leq i \leq p$ . Consequently, one has  $A^1(X, K_2) = E_p^{1,-2}(X)$ .

Moreover, by the result [11, Theorem 3.4] of A. Merkurjev, for any smooth variety  $X$ , every prime divisor  $l$  of the order of the differential  $\delta_r$  ending in  $E_r^{p+1,-p-1}(X)$  is such that  $l - 1$  divides  $r - 1$ . Therefore, for any prime  $p$  and  $2 \leq r \leq p - 1$ , the differential  $\delta_r$  is of prime to  $p$  order. Assume furthermore that  $X$  is a  $\mathbf{SL}_1(A)$ -torsor, with  $A$  of prime degree  $p$ . Since  $p \cdot \mathrm{CH}^{p+1}(X) = 0$  (see Remark 2.1), one deduce that, for  $2 \leq r \leq p - 1$ , the differential  $\delta_r$  is trivial. Consequently, one has  $\mathrm{CH}^{p+1}(X) = E_p^{p+1,-p-1}(X)$ .

Therefore, for  $X$  a  $\mathbf{SL}_1(A)$ -torsor, with  $A$  of prime degree  $p$ , the differential  $\delta_p$  in the BGQ-structure is a homomorphism

$$\delta : A^1(X, K_2) \rightarrow \mathrm{CH}^{p+1}(X).$$

**Remark 2.2.** Let  $X$  be a principal homogeneous space for a semisimple group  $G$ . By [6, Part II, Example 4.3.3 and Corollary 5.4], one has  $E_2^{0,-1}(X) = A^0(X, K_1) = F^\times$  and the composition  $F^\times = K_1(F) \rightarrow K_1(X) \rightarrow A^0(X, K_1)$  of the pullback of the structural morphism with the inclusions

$$K_1^{(0/1)}(X) = E_\infty^{0,-1}(X) \subset \cdots \subset E_3^{0,-1}(X) \subset E_2^{0,-1}(X)$$

given by the BGQ spectral sequence, is the identity. Therefore, for any  $i \geq 2$ , the differential starting from  $E_i^{0,-1}(X)$  is zero, i.e for any  $i \geq 2$ , one has

$$E_i^{i,-i}(X) = \tau^{i/i+1}(X).$$

In particular, for  $X$  a  $\mathbf{SL}_1(A)$ -torsor, with  $A$  of prime degree  $p$ , one has  $E_{p+1}^{p+1,-p-1}(X) = 0$ , i.e the differential  $\delta : A^1(X, K_2) \rightarrow \mathrm{CH}^{p+1}(X)$  is surjective.

**2.3. On the group  $A^1(X, K_2)$ .** The proof in the next section will use the work of A. Merkurjev on the *Rost invariant* of simply connected algebraic groups (see [6, Part II]). Let  $X$  be a  $\mathbf{SL}_1(A)$ -torsor over  $F$ . The group  $A^1(X_{F(X)}, K_2)$  is infinite cyclic with generator  $q$  and isomorphic to  $A^1(\mathbf{SL}_n, K_2)$  under restriction (where  $n = \deg(A)$ ). Furthermore, the restriction map  $r : A^1(X, K_2) \rightarrow A^1(X_{F(X)}, K_2)$  is injective with finite cokernel of same order as the element  $R_{\mathbf{SL}_1(A)}(X)$ , where

$$R_{\mathbf{SL}_1(A)} : H^1(F, \mathbf{SL}_1(A)) \rightarrow H^3(F, \mathbb{Q}/\mathbb{Z}(2))$$

is the Rost invariant of  $\mathbf{SL}_1(A)$  (see [6, Theorem 9.10]). Moreover, the homomorphism  $R_{\mathbf{SL}_1(A)}$  is of order  $\exp(A)$  by [6, Theorem 11.5].

If  $\mathrm{char}(F) = l$  is prime then the modulo  $l$  component  $H^3(F, \mathbb{Z}/l\mathbb{Z}(2))$  of the Galois cohomology group  $H^3(F, \mathbb{Q}/\mathbb{Z}(2))$  is the group  $H_l^3(F)$  defined by K. Kato in [10] by means of logarithmic differential forms.

## 3. PROOF OF THE RESULT

In this section, we prove the result of this note.

**Theorem 3.1.** *Let  $A$  be a central simple algebra of prime degree  $p$  over a field  $F$  and let  $X$  be a  $\mathbf{SL}_1(A)$ -torsor. Then*

- (i) *for any equidimensional  $F$ -variety  $Y$ , the change of field homomorphism*

$$\mathrm{CH}(Y) \rightarrow \mathrm{CH}(Y_{F(X)}),$$

*where  $\mathrm{CH}$  is the integral Chow group, is surjective in codimension  $< p + 1$ .*

- (ii) *it is also surjective in codimension  $p + 1$  for a given  $Y$  provided that the variety  $X_{F(\zeta)}$  does not have any closed point of prime to  $p$  degree for each generic point  $\zeta \in Y$ .*

*Proof.* We use notations and materials introduced in the previous section. One can assume that  $X$  does not have any rational point over  $F$  (or equivalently  $X$  does not have any closed point of prime to  $p$  degree, by the result [1, Theorem 3.3] of J. Black), if else there is nothing to prove. Note that in this situation, the central simple algebra  $A$  is necessarily a division algebra. We recall that conclusion (i) has already been proved (see Remark 2.1). According to Proposition 1.2, it suffices to show that  $\mathrm{CH}^{p+1}(X_{F(\zeta)}) = 0$  for each generic point  $\zeta \in Y$  to get conclusion (ii). Since  $X_{F(\zeta)}$  does not have any closed point of prime to  $p$  degree, it is enough to prove that  $\mathrm{CH}^{p+1}(X) = 0$ .

Assume on the contrary that  $\mathrm{CH}^{p+1}(X) \neq 0$ . Then  $\delta : A^1(X, K_2) \rightarrow \mathrm{CH}^{p+1}(X)$  is nonzero (since  $\delta$  is surjective by Remark 2.2), i.e.  $E_{p+1}^{1,-2}(X)$  is strictly included in  $E_p^{1,-2}(X) = A^1(X, K_2)$ . We claim that this implies that, by denoting as  $q_X$  the generator of  $A^1(X, K_2)$ , one has  $r(q_X) = q$ . Indeed, otherwise one has  $r(q_X) = p \cdot q$  by §2.3. Consecutively, by denoting as  $c$  the corestriction morphism  $A^1(\mathbf{SL}_p, K_2) \rightarrow A^1(X, K_2)$ , for any  $i \geq 2$ , one has  $c(E_i^{1,-2}(\mathbf{SL}_p)) = c(A^1(\mathbf{SL}_p, K_2)) = A^1(X, K_2)$  (where the first identity is due to  $\mathrm{CH}^i(\mathbf{SL}_p) = 0$  for any  $i \geq 2$ ). In particular, one has  $E_p^{1,-2}(X) = c(E_{p+1}^{1,-2}(\mathbf{SL}_p)) \subset E_{p+1}^{1,-2}(X)$ , which is a contradiction.

Therefore, we have shown that under the assumption  $\mathrm{CH}^{p+1}(X) \neq 0$ , the generator  $q$  of  $A^1(X_{F(X)}, K_2)$  is rational. Then it follows that the generator  $g$  of  $\mathrm{CH}^{p+1}(X_{F(X)})$  is also rational.

However, since  $A_{F(X)}$  is a still a division algebra, by [9, Theorem 7.2 and Theorem 8.2], the cycle  $g^{p-1}$  in  $\mathrm{CH}_0(\mathbf{SL}_1(A_{F(X)}))$  is nonzero and the latter group is cyclic of order  $p$  generated by the class of the identity of  $\mathbf{SL}_1(A_{F(X)})$ . Thus, the degree of the rational cycle  $g^{p-1}$  is prime to  $p$ .

It follows that  $X$  has a closed point of prime to  $p$  degree, which is a contradiction.

The Theorem is proved.  $\square$

**Remark 3.2.** The end of the above proof shows in particular that for a division algebra  $A$  of prime degree  $p$  over a field  $F$ , the kernel of the Rost invariant  $R_{\mathbf{SL}_1(A)}$  is trivial. This is already contained in the result [12, Theorem 12.2] of A. Merkurjev and A. Suslin under the assumption  $\mathrm{char}(F) \neq p$ . Indeed, let  $\xi \in H^1(F, \mathbf{SL}_1(A))$  and let  $X$  be the associated  $\mathbf{SL}_1(A)$ -torsor. Assume that  $R_{\mathbf{SL}_1(A)}(\xi)$  is trivial. It follows then by §2.3 that the generator of  $A^1(X_{F(X)}, K_2)$  is rational. As we have seen in the above proof, this implies that  $X$  has a rational point over  $F$ , i.e the cocycle  $\xi$  is trivial.

Note also that for a division algebra  $A$  of prime degree  $p$  over a field  $F$ , the Rost invariant  $R_{\mathbf{SL}_1(A)}$  coincides, up to sign, with the normalized invariant given by the cup product  $[A] \cup (c) \in H^3(F, \mathbb{Z}/p\mathbb{Z}(2))$  for any class  $c \in \mathrm{Nrd}(A^\times)$ , where  $[A]$  is the class of the algebra  $A$  in the Brauer group  $\mathrm{Br}(F)$ , see [6, §11].

#### 4. EXCEPTIONAL PROJECTIVE HOMOGENEOUS VARIETIES

In this section, we describe how Theorem 1.1 implies a similar version of it for projective homogeneous varieties under a group of type  $F_4$  or  $E_8$ . Namely, we give an alternative proof of Theorem 4.1 below. The following proof requires the characteristic of the base field to be different from  $p$ , with  $p = 3$  when  $G$  is of type  $F_4$  and  $p = 5$  when  $G$  is of type  $E_8$ , although the original result [4, Theorem 1.1] is valid for arbitrary characteristic.

Let  $X$  be a nonsplit  $\mathbf{SL}_1(A)$ -torsor over a field  $F$ , with  $A$  a division algebra of prime degree  $p$ . There exists a smooth compactification  $\tilde{X}$  of  $X$  such that the Chow motive  $\mathcal{M}(\tilde{X}, \mathbb{Z}/p\mathbb{Z})$  decomposes as a direct sum  $\mathcal{R}_p \oplus N$ , where  $\mathcal{R}_p$  is the indecomposable Rost motive associated with the symbol  $[A] \cup (c) \in H^3(F, \mathbb{Z}/p\mathbb{Z}(2))$ , with  $c \in F^\times \setminus \mathrm{Nrd}(A^\times)$  giving  $X$ , see [9, Theorem 1.1]. Note that the projective variety  $\tilde{X}$  is a *norm variety* of  $s$ .

**Theorem 4.1.** *Let  $G$  be a linear algebraic group of type  $F_4$  or  $E_8$  over a field  $F$  of characteristic different from  $p$ , with  $p = 3$  when  $G$  is of type  $F_4$  and  $p = 5$  when  $G$  is of type  $E_8$ , and let  $X'$  be a projective homogeneous  $G$ -variety. For any equidimensional variety  $Y$ , the change of field homomorphism*

$$\mathrm{Ch}(Y) \rightarrow \mathrm{Ch}(Y_{F(X')}),$$

where  $\mathrm{Ch}$  is the Chow group modulo  $p$ , is surjective in codimension  $< p + 1$ .

It is also surjective in codimension  $p + 1$  for a given  $Y$  provided that  $1 \notin \deg \mathrm{Ch}_0(X'_{F(\zeta)})$  for each generic point  $\zeta \in Y$ .

*Proof.* Since the  $F$ -variety  $X'$  is  $A$ -trivial in the sense of [8, Definition 2.3], one can assume that  $G$  has no splitting field of degree coprime to  $p$ . Indeed, otherwise  $1 \in \deg \mathrm{Ch}_0(X')$  by corestriction and this implies that  $\mathrm{Ch}(Y) \rightarrow \mathrm{Ch}(Y_{F(X')})$  is an isomorphism in any codimension by  $A$ -triviality, see [8, Lemma 2.9].

Let us now write  $G = {}_\xi G_0$  for a nontrivial cocycle  $\xi \in H^1(F, G_0)$ , with  $G_0$  a split group of the same type as  $G$ . Then the motive  $\mathcal{R}_p(G)$  living on the Chow motive (with coefficients in  $\mathbb{Z}/p\mathbb{Z}$ ) of  $X'$  given in [15, Theorem 5.17] is the Rost motive of the symbol  $R_{G_0,p}(\xi) = [A] \cup (c) \in H^3(F, \mathbb{Z}/p\mathbb{Z}(2))$ , where  $R_{G_0,p}$  is the modulo  $p$  component of the Rost invariant  $R_{G_0}$ ,  $A$  is a division algebra of degree  $p$  and  $c \in F^\times \setminus \mathrm{Nrd}(A^\times)$  – see [13, §4] and [5, §14] (here the assumption  $\mathrm{char}(F) \neq p$  is needed).

Let us denote as  $X$  the nonsplit  $\mathbf{SL}_1(A)$ -torsor over  $F$  associated with  $c$  and as  $\tilde{X}$  its smooth compactification. We claim that  $X'$  has a closed point of prime to  $p$  degree over  $F(\tilde{X})$  and vice versa.

Indeed, since  $\tilde{X}$  is a norm variety for  $[A] \cup (c)$ , the motive  $\mathcal{R}_p(G)$  decomposes as a sum of Tate motives over  $F(\tilde{X})$ . Therefore, the group  $G_{F(\tilde{X})}$  is split by an extension of degree coprime to  $p$  and it follows that  $X'$  has a closed point of prime to  $p$  degree over  $F(\tilde{X})$  (this is more generally true for any extension  $L/F$  over which  $\tilde{X}$  has a closed point of prime to  $p$  degree). Moreover, the motive  $\mathcal{R}_p(G)$  decomposes as a sum of Tate motives

over  $F(X')$  because  $G$  is split by  $F(X')$ . Consequently,  $\tilde{X}$  has a closed point of prime to  $p$  degree over  $F(X')$ .

It follows then (note that  $\tilde{X}$  is  $A$ -trivial by [8, Example 5.7]) that the right and the bottom homomorphisms in the commutative square

$$\begin{array}{ccc} \mathrm{Ch}(Y) & \longrightarrow & \mathrm{Ch}(Y_{F(X')}) \\ \downarrow & & \downarrow \\ \mathrm{Ch}(Y_{F(\tilde{X})}) & \longrightarrow & \mathrm{Ch}(Y_{F(\tilde{X} \times X')}) \end{array}$$

are isomorphisms. Since  $F(\tilde{X}) = F(X)$ , Theorem 4.1 is now a direct consequence of Theorem 1.1.  $\square$

The following was pointed out to me by Philippe Gille.

**Remark 4.2.** Let  $G_0$  a split group of type  $E_8$  over a 5-special field  $F$  (i.e  $F$  has no proper extension of degree coprime to 5) of characteristic  $\neq 5$ . The above proof gives rise to a new argument for the triviality of the kernel of the Rost invariant modulo 5

$$H^1(F, G_0) \rightarrow H^3(F, \mathbb{Z}/5\mathbb{Z}(2)).$$

This result is originally due to Vladimir Chernousov (under the assumption  $\mathrm{char}(F) \neq 2, 3, 5$ , see [2, Theorem]).

Indeed, since  $F$  is 5-special, for any nontrivial cocycle  $\xi \in H^1(F, G_0)$ , the group  ${}_{\xi}G_0$  has no splitting field of degree coprime to 5. Then, as we have seen in the proof, there is a division algebra  $A$  of degree 5 such that  $R_{G_0,5}(\xi)$  is equal to a symbol  $[A] \cup (c)$  associated with a nonsplit  $\mathbf{SL}_1(A)$ -torsor  $X$ . The injectivity of  $R_{G_0,5}$  follows now from Remark 3.2.

## REFERENCES

- [1] BLACK, J. Zero cycles of degree one on principal homogeneous spaces. *Journal of Algebra* 335, 1 (2011), 232–246.
- [2] CHERNOUSOV, V. Remark on the (mod 5)-invariant of serre for groups of type  $E_8$ . *Math. Notes*. 56, 1 (1994), 730–733.
- [3] ELMAN, R., KARPENKO, N., AND MERKURJEV, A. *The algebraic and geometric theory of quadratic forms*, vol. 56 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2008.
- [4] FINO, R. Rationality of cycles over function field of exceptional projective homogeneous varieties. *J. Ramanujan Math. Soc.* 29, 1 (2014), 119–132.
- [5] GARIBALDI, S. Cohomological invariants: exceptional groups and spin groups. *Memoirs Amer. Math. Soc.* 200, 937 (2009). With an appendix by Detlev W. Hoffmann.
- [6] GARIBALDI, S., MERKURJEV, A., AND SERRE, J.-P. *Cohomological invariants in Galois cohomology*, vol. 28 of *University Lecture Series*. Amer. Math. Soc., 2003.
- [7] GILLE, P., AND SZAMUELY, T. *Central simple algebras and Galois cohomology*, vol. 101 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2006.
- [8] KARPENKO, N., AND MERKURJEV, A. On standard norm varieties. *Ann. Sci. Ec. Norm. Sup. (4)* 46, fascicule 1 (2013), 175–214.
- [9] KARPENKO, N., AND MERKURJEV, A. Motivic decomposition of compactifications of certain group varieties. [www.math.uni-bielefeld.de/lag/man/528.html](http://www.math.uni-bielefeld.de/lag/man/528.html) (22 Feb 2014), 18 pages. (Preprint).
- [10] KATO, K. Galois cohomology of complete discrete valuation fields. *Lect. Notes in Math.*, 967 (1982), 215–238.

- [11] MERKURJEV, A. Adams operations and the Brown-Gersten-Quillen spectral sequence. In *Quadratic forms, linear algebraic groups, and cohomology*, vol. 18 of *Dev.Math.* Springer, New York, 2010, pp. 305–313.
- [12] MERKURJEV, A., AND SUSLIN, A. K-cohomology of Severi-Brauer varieties and the norm residue homomorphism. *Izv. Akad. Nauk SSSR*, 46 (1982), 1011–1046.
- [13] NIKOLENKO, S., SEMENOV, N., AND ZAINOULLINE, K. Motivic decomposition of anisotropic varieties of type  $F_4$  into generalized Rost motives. *J. of K-theory* 3, 1 (2009), 114–124.
- [14] PANIN, I. Splitting principle and K-theory of simply connected semisimple algebraic groups. *Algebra i Analiz* 10, 1 (1998), 88–131. translation in *St. Petersburg Math. J.* 10 (1999), n°1, 69–101.
- [15] PETROV, V., SEMENOV, N., AND ZAINOULLINE, K.  $J$ -invariant of linear algebraic groups. *Ann. Sci. Ec. Norm. Sup. (4)* 41, 6 (2008), 1023–1053.
- [16] QUILLLEN, D. Higher algebraic K-theory. I. (1973). vol. 341 of *Lecture Notes in Math.*, pp. 85–147.
- [17] SUSLIN, A. A. K-Theory and  $\mathcal{K}$ -cohomology of certain group varieties. *Advances in Soviet Mathematics* 4 (1991), 53–74.

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